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Annals of Pure and Applied Logic 137 (2006) 299–316
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Regular universes and formal spaces

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Available online 17 June 2005

Abstract

We present an alternative solution to the problem of inductive generation of covers in formal topology by using a restricted form of type universes. These universes are at the same time constructive analogues of regular cardinals and sets of infinitary formulae. The technique of regular universes is also used to construct canonical positivity predicates for inductively generated covers.

MSC: 03F65; 54A05

Keywords: Point-free topology; Predicativity; Type theory

1. Introduction

An essential feature of formal topology is its modest assumption on the meta-theory. It may be based on a predicatively constructive foundation such as Martin-Löf type theory, or Aczel–Myhill set theory. This is in contrast to standard locale theory [7] which takes off from the notion of complete lattice — an essentially impredicative construction. Locale theory may however be developed in a topos [9] or in similar contexts with a comprehension principle. For formal topology it is thus of importance to try to replace various impredicative constructions, such as very big intersection sets, by inductive definitions. Coquand et al. [4] penetrate the problem of inductively generating formal topologies. Solutions to the problem are provided for set-presented topologies with positivity predicate, using an analysis of derivation trees for covers.

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doi:10.1016/j.apal.2005.05.025

In this paper we provide a different solution which is less proof-theoretic in spirit and which seems easy to extend to other versions of formal spaces. The idea is to identify a more restricted class of formal spaces which serve as a basis for set-presented topologies. The cover rules, particularly the transitivity rule, may then more easily be seen to be inductive. The covering sets are restricted to a *regular power set*, so called because its truth conditions lie in a regular type universe, which in turn has a direct analogue in the notion of regular cardinal. Another advantage of studying these more restrictive topologies is that they give a tighter control of the logical complexity of notions like cover and point (cf. [18]). In the classical setting, the effective formal spaces, as developed in the 1990 PhD-thesis [22] of Inger Sigstam (supervised by Viggo Stoltenberg-Hansen) stands out as one particular interesting subclass (cf. the articles by Sigstam [23] and Sigstam and Stoltenberg-Hansen [24]). Via realisability models of type theory, or effective toposes, it should be possible to relate the effective approach to the constructive approach.

One motivation for writing this paper was to suggest some alternative proof techniques, which are also easy to formalize in type theory. In [19] a formal proof of minimality is given. We have thus rephrased many known results and reconstructed proofs into this language, whose exact attribution can be found in the papers of Coquand et al. [4], Sambin [20,21] and Coquand [3]. We also believe that the paper suggests that regular universes could be useful for constructivizing parts of locale theory, or categorical model theory, that depends on the classical notion of regular cardinal [11,12].

2. Formal topologies — sites

Let X be a set. For any binary relation $\leq$ on X define the *formal intersection* of subsets $U, V \subseteq X$ with respect to the relation as

$$U \wedge V = \{x \in X : (\exists u \in U)\, x \leq u \;\&\; (\exists v \in V)\, x \leq v\}. \tag{1}$$

By considering singletons it is easy to see that $\wedge$ satisfies the inclusion $W \subseteq W \wedge W$ if, and only if, the relation $\leq$ is reflexive. For a reflexive $\leq$, the operation $\wedge$ is associative if, and only if, $\leq$ is transitive.

Definition 2.1. Let X be a set, and let $\lhd$ be a relation between elements of X and subsets of X, i.e. $\lhd \subseteq X \times \mathcal{P}(X)$. Such a relation is called an *abstract cover relation* on X. Extend $\lhd$ to a relation between subsets of X:

$$U \lhd V \Longleftrightarrow_{\mathrm{def}} (\forall p \in U)\, p \lhd V.$$

The relation can also be considered as a binary relation on X by letting

$$a \lhd b \Longleftrightarrow_{\mathrm{def}} a \lhd \{b\}.$$

The formal intersection associated with this relation is denoted $\mathit{\Delta}$.

We shall use the terms *formal space* and *formal topology* almost interchangeably, with preference of the former when points play an important role. There is a variety of definitions of formal topologies. A formal topology generally consists of a set X, of so-called *basic opens* or *neighbourhoods*, together with an abstract cover relation on the neighbourhoods satisfying certain conditions. There may also be additional structure, such as a positivity predicate. In this paper we shall consider a version based on preorders [24] which can easily be compared to Grothendieck topologies.

Definition 2.2. The following defines various properties of an abstract cover relation $\lhd$

(R) $a \in U$ implies $a \lhd U$,
(T) $a \lhd U$, $U \lhd V$ implies $a \lhd V$,
(L) $a \lhd U$, $a \lhd V$ implies $a \lhd U \mathbin{\Delta} V$.

Here R is short for reflexivity, T for transitivity and L for localization.

This version of the localisation axiom (L) has the drawback that the intersection $\cap_i \lhd_i$ of a set of covering relations need not be a covering relation (cf. discussion in [4]). For this reason the formal intersection is defined in terms of a fixed preorder.

Definition 2.3. Let $X = (X, \leq)$ be a preorder, i.e. a set with a reflexive and transitive relation. A *formal topology* (with preorder) is such an X together with an abstract cover relation $\lhd$ on X satisfying the axioms (R), (T) and

(QL) $a \lhd U$, $a \lhd V$ implies $a \lhd U \wedge V$,
(QE) $a \leq b$ implies $a \lhd \{b\}$.

Here $\wedge$ is the formal intersection associated with $\leq$.

A subset $\alpha \subseteq X$ is *filtering* with respect to $\leq$ if for any $x \in \alpha$, $y \in \alpha$ there exists $z \in \alpha$ with $z \leq x$ and $z \leq y$.

A *point* is an inhabited subset $\alpha \subseteq X$ which is filtering with respect to $\leq$, and such that $U \cap \alpha$ is inhabited whenever $a \lhd U$ for some $a \in \alpha$.

A subset $P \subseteq X$ is called a *positivity candidate* for the formal space X, if for each $a \in X$ and every $U \subseteq X$

$$(a \in P \Rightarrow a \lhd U) \Longrightarrow a \lhd U.$$

2.1. Relation to sites

We indicate the relation of this notion of formal topology to *Grothendieck topology* or *site*. (See Mac Lane and Moerdijk [10] for background.) The following notation is used. Define $U \cdot a = U \wedge \{a\}$, the *restriction of U to a*. Let $U_\downarrow = \{x \in X : (\exists a \in U)\, a \leq x\}$. Note that $U_\downarrow = U \wedge U$.

We leave the verification of the following as a straightforward exercise.

Proposition 2.4. *For a formal topology* $(X, \leq, \lhd)$

(i) $a \lhd U$ *iff* $a \lhd U_\downarrow$,
(ii) $a \lhd U_\downarrow$ *iff* $a \lhd U \cdot a$,
(iii) $a \lhd \{a\}_\downarrow$,
(iv) $a \lhd U$ *and* $b \leq a$ *implies* $b \lhd U \cdot b$,
(v) *if* $a \lhd U$, *and* $(\forall b \in U)\, b \lhd V \cdot b$, *then* $a \lhd V$. $\square$

A subset $U \subseteq X$ is a *sieve on a* if $x \leq a$ for all $x \in U$, and

$$y \in X, x \in U, y \leq x \Longrightarrow y \in U.$$

Clearly $U \cdot a$ is a sieve on a for *any* $U \subseteq X$. In particular, $\{a\}_\downarrow$ is a sieve on a. Note $U \subseteq X$ is a sieve on a iff

$$U \cdot a = U.$$

It is also easy to see that an intersection of two sieves on a is again a sieve on a. Properties (iii)–(v) of Proposition 2.4 indicate how the covering relation can be restricted to sieves.

Definition 2.5. A *Grothendieck formal topology* consists of a preorder $(X, \leq)$ and a cover relation

$$\lhd \subseteq \{(a, U) : a \in X, U \subseteq X \text{ sieve on } a\}$$

that satisfies

(G1) $a \lhd \{a\}_\downarrow$,
(G2) $a \lhd U$ and $b \leq a$ implies $b \lhd U \cdot b$,
(G3) if $a \lhd U$, and V is a sieve on a with $(\forall b \in U)\, b \lhd V \cdot b$, then $a \lhd V$. $\square$

Lemma 2.6. *For a Grothendieck formal topology* $(X, \leq, \lhd)$, *and sieves* U *and* V *on* $a \in X$

(i) $a \lhd U$, $U \subseteq V$ *implies* $a \lhd V$,
(ii) $a \lhd U$, $a \lhd V$ *implies* $a \lhd U \cap V$. $\square$

For any Grothendieck formal topology $(X, \leq, \lhd)$ we define a $(X, \leq)$-based formal topology $(X, \leq, \lhd^*)$ by

$$a \lhd^* U \Longleftrightarrow_{\text{def}} a \lhd U \cdot a.$$

Proposition 2.7. $(X, \leq, \lhd^*)$ *is a formal topology.* $\square$

3. Regular universes in type theory

In the classical theory of inductive definitions regular cardinals play an important role as closure stages for positive operators. A constructively useful version of this notion turns out to be *regular set* [1]. In type theory a corresponding notion is that of *regular type universe.*

We first briefly recall some conventions for formalizations in Martin-Löf type theory. In this type theory there are two classes of types, one smaller called the *sets* which consists of types that (can) have elimination rules, and one larger just called *types*. The collection of sets is a type, called Set, but is itself not a set. Also the collection $X \to \text{Set}$ of families of sets over a set X is a type. According to the propositions-as-types principle, this is also the "power" of X, $\mathcal{P}(X)$. It is not a set, so we cannot universally or existentially quantify over it when forming new subsets. In constructive mathematics (Bishop and Bridges [2]) every set X is usually equipped with a defined equivalence $=_X$, which in type-theoretic formalization is an element of $\mathcal{P}(X \times X)$. The pair $(X, =_X)$ is then a *set with equality*. We shall abbreviate this to simply *set* except in the present section. Bishop calls the first component a *preset*, and we shall use this term when we need to make the distinction later. Quantification over a set S is usually written using ":" $(\forall x : S)\cdots$ or $(\exists x : S)\cdots$, whereas quantification over a fixed subset $U \subseteq S$ is written $(\forall x \in U)\cdots$, meaning $(\forall x : S)(x \in U \Rightarrow \cdots)$, and $(\exists x \in U)\cdots$, meaning $(\exists x : S)(x \in U \wedge \cdots)$.

Let $B : A \to \mathrm{Set}$ be a family of sets indexed by $A : \mathrm{Set}$. Then we assume $\mathcal{U} = \mathcal{U}(B)$ and $\mathcal{T} = \mathcal{T}(B)$ is a family of sets satisfying

$$\frac{a : A}{\varepsilon(a) : \mathcal{U}}, \qquad \mathcal{T}(\varepsilon(a)) = B(a) \tag{2}$$

and

$$\frac{a : \mathcal{U} \qquad b : \mathcal{T}(a) \to \mathcal{U}}{\sigma(a, b) : \mathcal{U}}, \qquad \mathcal{T}(\sigma(a, b)) = (\Sigma x : \mathcal{T}(a))\mathcal{T}(b(x)). \tag{3}$$

It is thus a universe closed under Σ which includes the family B of sets. Note the following formal similarity between $\mathcal{U}$ and a regular cardinal κ: if $\kappa_i < \kappa$ for all $i \in I$, where $|I| < \kappa$, then $\cup_{i \in I} \kappa_i < \kappa$. Any such universe is called a *regular universe enclosing* B. Here ε is not necessarily a constructor. An ordinary type-theoretic universe [14] may then be considered as a regular universe enclosing many different B. The minimal $\mathcal{U}, \mathcal{T}$ defined by the introduction rules (2) and (3) is called the *canonical regular universe* enclosing B. This universe could also be considered as the set of semi-decidable sets relative to the family B. Informally, there is a type-theoretic isomorphism

$$(\Sigma z : (\Sigma x : A_1)A_2(x))A_3(z) \cong (\Sigma x : A_1)(\Sigma y : A_2(x))A_3(\langle x, y\rangle).$$

which allows us to rewrite types in normal form. In the universe $\mathcal{U}(B)$, which encloses the family $B(x)$ $(x : A)$, any type may be written, up to isomorphism, in one of the two forms

$$B(s) \qquad\qquad (\Sigma y : B(t))A(y)$$

where $A(y)$ is again of this form.

Consider an arbitrary regular universe $\mathcal{U}, \mathcal{T}$. We note that the set of propositions in $\mathcal{U}$ is closed under conjunction ($a \,\&\, b = \sigma(a, (x)b)$) and existential quantification over sets in $\mathcal{U}$: $(\exists x \in s)p(x) = \sigma(s, p)$. One should not underestimate the expressive power of propositions in a regular universe containing infinite sets, since the σ-construction may code infinite disjunctions over such sets. This can be seen as a constructive version of infinitary coherent logic [13].

Define the *regular power set of* X corresponding to the universe $\mathcal{U}$ as

$$\mathcal{R}_{\mathcal{U}}(X) =_{\mathrm{def}} X \to \mathcal{U}.$$

The universe $\mathcal{U}$ is then the set of admissible *membership conditions* for the subsets of X. When $\mathcal{U}$ is clear from the context we shall sometimes drop the subscript. Suppose $X = (X, =_X)$ is a set X with an equivalence relation $=_X$. A subset $S \subseteq X$ is *extensional (with respect to* $=_X$*)* if $a =_X b$ and $a \in S$ implies $b \in S$. The *extensional power set* $\mathcal{R}^e(X)$ consists of pairs (S, r), where S is an extensional subset $S : \mathcal{R}(X)$ and r is a proof object for this fact. In order that subsets of a structure of interest be definable in such power sets, we may have to require that its associated universe includes certain data of the structure. We say that $\mathcal{U}$ *contains a set* X if there is some $x_0 : \mathcal{U}$ with $X = \mathcal{T}(x_0)$. A relation (or subset) $R \subseteq X$ is *represented in* $\mathcal{U}$ if there exists a *characteristic function into* $\mathcal{U}$, i.e. some $\chi_R : X \to \mathcal{U}$ such that for all $x : X$

$$R(x) \Longleftrightarrow \mathcal{T}(\chi_R(x)).$$

More generally, a structure $(X; \{R_i\}_{i \in I})$ with relations $R_i \subseteq X^{n_i}$ is called $\mathcal{U}$*-small*, if X and each preset X^{n_i} is contained in $\mathcal{U}$, and each relation $R_i \subseteq X^{n_i}$ is represented in $\mathcal{U}$. For instance, that an arbitrary binary relation $\leq \subseteq X \times X$ is represented in the universe means then, naturally, that there is a characteristic function $\chi_{(\leq)} : X \times X \to \mathcal{U}$ such that

$$a \leq b \Longleftrightarrow \mathcal{T}(\chi_{(\leq)}(a, b)). \tag{4}$$

A set with equality $X = (X, =_X)$ is $\mathcal{U}$-small, if it is $\mathcal{U}$-small as structure. Note that for singletons

$$\{x\} = ((\lambda y : X)(\chi_{(=_X)}(x, y)), r) : \mathcal{R}^e(X),$$

where r is a proof of extensionality.

Example 3.1. Let $S = (S, =_S)$ be a set with a preorder $\leq \subseteq S \times S$, which is extensional with respect to $=_S$. Suppose that $\mathcal{U}$ is a universe such that the structure $S = (S, =_S, \leq)$ is $\mathcal{U}$-small. If $U, V : \mathcal{R}_{\mathcal{U}}(S)$, then the formal intersection $U \wedge V : \mathcal{R}_{\mathcal{U}}(S)$. This is seen by noting that the definition (1) can be coded as follows: $U \wedge V = (\lambda x : S)\, p(x, U, V)$ where $p(x, U, V) = q(x, U) \,\&\, q(x, V)$ and

$$q(x, W) = (\exists y : S)\, (W(y) \,\&\, \chi_{(\leq)}(x, y)).$$

This is not more difficult than finding the formal expression in first-order logic, and we shall not pursue such coding explicitly. Since $\leq$ respects $=_X$ it is easy to see that if U, V are in $\mathcal{R}^e_{\mathcal{U}}(S)$, then so is $U \wedge V$. □

By similar considerations it is easy to see that regular power sets with membership conditions in $\mathcal{U}$ are closed under unions indexed by sets in $\mathcal{U}$, i.e. $\cup_{i:I} V_i$, where $I = \mathcal{T}(c)$ for some $c : \mathcal{U}$.

Example 3.2. Consider the problem of finding the canonical regular universe $\mathcal{U}$ such that the structure $(S, \leq)$ is $\mathcal{U}$-small. We are to construct a family of types $B : A \to \mathrm{Set}$, so that $\mathcal{U} = \mathcal{U}(B)$. This involves another simple coding technique. Let $N_1 = \{0_1\}$ be a one element set and $+$ disjoint union. First take

$$A = N_1 + (S \times S) + (S \times S)$$

and let $\mathrm{i}_S = \mathrm{i}_1$, $\mathrm{i}_{(=_X)} = \mathrm{i}_2$, $\mathrm{i}_{(\leq)} = \mathrm{i}_3$ denote the respective injection functions (cf. [14]). Then define B by cases

$$\begin{aligned} B(\mathrm{i}_S(0_1)) &= S, \\ B(\mathrm{i}_{(=_X)}(a, b)) &= (a =_X b), \\ B(\mathrm{i}_{(\leq)}(a, b)) &= (a \leq b). \end{aligned}$$

Here $(a =_X b)$ and $(a \leq b)$ are regarded as types. Notice that the characteristic function for $\leq$ in $\mathcal{U}$ now becomes

$$\chi_{(\leq)}(a, b) = \varepsilon(\mathrm{i}_{(\leq)}(a, b)),$$

since

$$\mathcal{T}(\chi_{(\leq)}(a, b)) = \mathcal{T}(\varepsilon(\mathrm{i}_{(\leq)}(a, b))) = B(\mathrm{i}_{(\leq)}(a, b)) = (a \leq b). \quad \square$$

Remark 3.3. Canonical regular universes are similar to the universe operators introduced in [17].

3.1. Collecting sets in a regular universe

Consider a proposition of the form

$$(\forall x : X)(x \in U \Rightarrow (\exists y : Y)P(x, y))$$

where $U \subseteq X$ and suppose that we want to collect the witnesses y. Type-theoretic choice is not directly applicable, so we form the sigma-set

$$\hat{U} = (\Sigma x : X)x \in U,$$

and then apply choice. Thus we get a choice function $f : \hat{U} \to Y$ such that

$$(\forall z : \hat{U})P(\pi_1(z), f(z)), \tag{5}$$

where $\pi_1(x, r) = x$ is the first projection. Now, if X is a type in the universe $\mathcal{U}$ and $U : \mathcal{R}_{\mathcal{U}}(X)$, we have that $\hat{U}$ belongs to the universe. We have thus proved a kind of *collection principle*. Consider now the case where $Y = \mathcal{R}_{\mathcal{U}}(X)$ and the property P is *monotone* on a subset $X_0 \subseteq X$ in the following sense. For all $x : X$ and all $S, T : \mathcal{R}_{\mathcal{U}}(X)$ the two conditions below are satisfied:

$$P(x, S) \Longrightarrow S \subseteq X_0, \tag{6}$$

$$S \subseteq T \subseteq X_0,\ P(x, S) \Longrightarrow P(x, T). \tag{7}$$

Form the union

$$V = \bigcup_{z:\hat{U}} f(z),$$

which is in $\mathcal{R}_{\mathcal{U}}(X)$. By (6) also $V \subseteq X_0$. Thus by (5) and monotonicity, we have

$$(\forall z : \hat{U})P(\pi_1(z), V).$$

Hence $P(x, V)$ for any $x \in U$. The above argument is also valid for extensional subsets. We have proved the following result which will be crucial later.

Theorem 3.4. *Suppose $X = (X, =_X)$ is a set with equality and that $\mathcal{U}$ is any (regular) universe that contains it. Let $\mathcal{R}^e(X) = \mathcal{R}^e_{\mathcal{U}}(X)$. Suppose that the property $P(x, U)$ ($x : X$, $U : \mathcal{R}^e(X)$) is monotone on $X_0 \subseteq X$ as in (6) and (7). Then for any $U : \mathcal{R}^e(X)$ with*

$$(\forall x \in U)(\exists W : \mathcal{R}^e(X))P(x, W)$$

there exists some $V : \mathcal{R}^e(X)$ *such that* $V \subseteq X_0$ *and*

$$(\forall x \in U) P(x, V). \quad \square$$

4. $\mathcal{U}$-formal topologies

We consider as in Section 2 formal topologies based on a preorder $(X, \leq)$ where X is equipped with an equivalence relation $=_X$. A regular universe $\mathcal{U}$ is said to be a *universe over* the preorder structure $(X, =_X, \leq)$, if this structure is $\mathcal{U}$-small. We will exclusively be interested in extensional notions here, so *set* means set with equality, unless otherwise indicated. From now on $\mathcal{R}_{\mathcal{U}}(X)$ denotes the extensional power set $\mathcal{R}^e_{\mathcal{U}}(X)$ with respect to the universe $\mathcal{U}$.

We now restrict the covers of formal topologies to regular power sets and obtain the following notion.

Definition 4.1. Let $X = (X, \leq)$ be a preorder. Let $\mathcal{U}$ be a regular universe over this structure, and $\mathcal{R}(X) = \mathcal{R}_{\mathcal{U}}(X)$.

A *$\mathcal{U}$-formal topology* or *formal topology relativised to* $\mathcal{U}$ is a triple $(X, \leq, \lhd)$ where the cover relation $\lhd \subseteq X \times \mathcal{R}(X)$ satisfies conditions (R), (T), (QL) and (QE).

A $\mathcal{U}$-point is any $\alpha : \mathcal{R}(X)$ which is a point with respect to the covers of the $\mathcal{U}$-formal topology.

A *positivity candidate* for a $\mathcal{U}$-formal space is defined similarly as for a formal space with the difference that the sets U vary only over $\mathcal{R}(X)$.

Note that by the definability considerations we went through in Section 3, the conditions (QL) and (QE) actually makes sense in the more restrictive setting. Analogously to effective formal spaces [24] the $\mathcal{U}$-points are generalised "semi-decidable" in the terminology of Section 3.

Let $X = (X, \leq, \lhd)$ be a $\mathcal{U}$-formal topology. Extend $\lhd$ to an abstract cover relation $\lhd^* \subseteq X \times \mathcal{P}(X)$ as follows:

$$a \lhd^* U \iff (\exists U' : \mathcal{R}(X)) \, [U' \subseteq U \;\&\; a \lhd U']. \tag{8}$$

Write $X^* = (X, \leq, \lhd^*)$ for the extension of X. Then $a \lhd^* U$ means that $a \lhd U'$ for some $U' \subseteq U$. For α, a $\mathcal{U}$-point, $a \in \alpha$ implies that $U' \cap \alpha \subseteq U \cap \alpha$ is inhabited. Hence α is also an ordinary point.

Theorem 4.2. *Let X be a $\mathcal{U}$-formal topology. Then:*

(i) *The extension X^* is a formal space.*
(ii) *Any $\mathcal{U}$-point of X is a point of X^*. Conversely, if α is a point of X^*, with $\alpha : \mathcal{R}(X)$, then α is a $\mathcal{U}$-point of X.*
(iii) *If $P : \mathcal{R}(X)$ is a positivity candidate for X, then it is a positivity candidate for X^* as well.*

Proof. Part (i): We check the conditions for a formal space.

(R): Suppose that $U \subseteq X$ and $x \in U$. Then $\{x\} \subseteq U$ and the singleton belongs to $\mathcal{R}(X)$.

(QE): Trivial.

(QL): Suppose $a \lhd^* U$ and $a \lhd^* V$. Thus $a \lhd U'$ and $a \lhd V'$ for some $U' \subseteq U$ and $V' \subseteq V$. Then by (QL) for the $\mathcal{U}$-formal topology $a \lhd U' \wedge V'$. But $U' \wedge V' \subseteq U \wedge V$ so we are done.

(T): This is the non-trivial case. Suppose $a \lhd^* U$ and $U \lhd^* V$. Thus there is $U' \subseteq U$ with $a \lhd U'$. Consider the property

$$P_V(x, W): \quad x \lhd W \;\&\; W \subseteq V$$

where $x : X$ and $W : \mathcal{R}(X)$. It is obviously monotone on V, since $\lhd$ is transitive. By definition,

$$x \lhd^* V \iff (\exists W : \mathcal{R}(X))\, P_V(x, W). \tag{9}$$

By $U' \subseteq U$, we have $x \lhd^* V$ for each $x \in U'$. Now Theorem 3.4 gives $V' : \mathcal{R}(X)$ so that $V' \subseteq V$ and $(\forall x \in U') P_V(x, V')$, i.e. $x \lhd V'$ for all $x \in U'$. Hence $U' \lhd V'$, and by transitivity for $\lhd$, $a \lhd V'$. We have proved $a \lhd^* V$.

Part (ii) was proved just before the statement of the theorem.

Part (iii): Let $P : \mathcal{R}(X)$ be a positivity candidate for X. Let $a \in X$ be fixed. Trivially, $a \in P \Rightarrow a \in \{a\} \cap P$. But the subset $\{a\} \cap P$ belongs to $\mathcal{R}(X)$, so $a \lhd \{a\} \cap P$. Hence also $a \lhd^* \{a\} \cap P$. Let $U \subseteq X$ and suppose that we know

$$a \in P \Rightarrow a \lhd^* U.$$

Thus $\{a\} \cap P \lhd^* U$, so by transitivity $a \lhd^* U$. Since a and U were arbitrary, we have shown that P is a positivity candidate for X^* as well. □

Remark 4.3. Formal topologies may contain partial (non-maximal) points. Such points need not be $\mathcal{U}$-points (cf. [18]).

Remark 4.4. The distinction between formal spaces and $\mathcal{U}$-formal space makes sense also in a topos-theoretic setting, using the collection maps of [15].

Set-presented (or set-based) formal topologies were introduced by P. Aczel. Here is an equivalent definition due to Martin-Löf and Sambin: $(X, \leq, \lhd)$ is *set-presented* if there are families of (pre)sets $I(a)$ $(a : X)$ and subsets $C(a, i)$ $(a : X, i : I(a))$ such that for all $a : X$ and all $U \subseteq X$,

$$a \lhd U \iff (\exists i : I(a))\, (C(a, i) \subseteq U). \tag{10}$$

Note that $(X, \leq, \lhd^*)$ of (8) is set-presented by $I(a) = (\Sigma V : \mathcal{R}(X))\,(a \lhd V)$ and $C(a, (V, r)) = V$. In fact, we have the following:

Theorem 4.5. *Consider a formal topology $(X, \leq, \lhd)$. It is set-presented if, and only if, for some regular universe $\mathcal{U}$, there is a $\mathcal{U}$-formal topology $(X, \leq, \lhd_1)$ so that $\lhd = \lhd_1^*$.*

Proof. $(\Leftarrow)$ was proved above.

$(\Rightarrow)$: Suppose that $\lhd$ is set-presented as in (10). Let $\mathcal{U}$ be so large that $(X, \leq)$ is $\mathcal{U}$-small, and the relations $C(a, i) \subseteq X$ are represented for all $a : X, i : I(a)$. We consider the regular power set $\mathcal{R}(X) = \mathcal{R}_{\mathcal{U}}(X)$. Define $\lhd_1 \subseteq X \times \mathcal{R}(X)$ by restricting $\lhd$ to

$\mathcal{R}(X)$ in the second argument. We claim that $\lhd = \lhd_1^*$. The inclusion $\supseteq$ is clear by transitivity. As for the reverse inclusion $\subseteq$: Suppose that $a \lhd U$. Then by (10), there is some $C(a, i) \subseteq U$. Note that $a \lhd C(a, i)$. Now since $C(a, i)$ belongs to $\mathcal{R}(X)$, $a \lhd_1 C(a, i)$. Thus $a \lhd_1^* U$. To see that $\lhd_1$ is a $\mathcal{U}$-topology is immediate, since we ensured that $\mathcal{R}(X)$ contains the singletons and is closed under $\wedge$. $\square$

5. Inductively generated cover relations

A look at the axioms (R), (T), (QL) and (QE) reveals that the intersection of a set of cover relations $\lhd_k \subseteq X \times \mathcal{P}(X)$

$$\bigcap_{k \in K} \lhd_k$$

is again a cover relation. Suppose that we are interested in the least cover containing some basic set of cover axioms

$$\text{(GE)} \quad g_i \lhd G_i \qquad (i : I).$$

In classical set theory, or topos theory, we may just take the intersection of all cover relations satisfying the first four axioms and (GE). However, this is a typically impredicative construction. So one may want to replace this construction by an inductive definition.

Using the rules for a formal topology as a method for inductive generation turns out to be the more complicated approach. As demonstrated by [4], it requires a great deal of care, in particular concerning the transitivity rule (T)

$$\frac{x \lhd U \qquad U \lhd V}{x \lhd V}.$$

This is not a predicatively acceptable introduction rule, when U varies freely over $\mathcal{P}(X)$. Indeed, they show that an impredicative principle ensues under such an assumption. Their solution for set-presented topologies is to let U only vary over basic cover axioms, and then prove (T) by an inductive argument similar to a cut-elimination proof. This procedure has to be done over again, if further closure rules are added (such as localisation). In the next section we show how to view the transitivity rule as an inductive generator using $\mathcal{U}$-formal topologies.

A less direct approach, but ultimately simpler and more elementary, has been pointed out by the referee of this paper. It is related to Johnstone's technique of *coverages* in locale theory. It is to view formal topologies as given by closure operators. For any subset $U \subseteq X$ inductively define what U covers, which will be a subset $A(U)$ of X. This may be done using standard techniques from inductive definitions in a first-order setting.

Let X^* be the set of finite sequences $[a_1, \ldots, a_n]$ of elements of X. Define a relation B between X and X^* as follows:

$$B(x, [a_1, \ldots, a_n]) \iff \bigwedge_{k=1}^{n} x \leq a_k.$$

For the empty list the right-hand side is, by convention, true for any x.

For any subset U of X, there is a smallest subset $A = A(U)$ on X such that

(F1) $U \subseteq A$,
(F2) $x \leq y$ and $y \in A \Longrightarrow x \in A$,
(F3) $x \leq g_i \wedge B(x, w) \wedge (\forall z, y \in X)[y \in G_i \wedge z \leq y \wedge B(z, w) \Rightarrow z \in A] \Rightarrow x \in A$.

Now define

$$a \lhd U \Longleftrightarrow_{\text{def}} x \in A(U).$$

Theorem 5.1. *The relation $\lhd$ is the smallest cover relation satisfying (GE) and so that $(X, \leq \lhd)$ is a formal topology.*

Proof. The proofs that $\lhd$ satisfies (R) and (QE) are straightforward. For (GE) note that $G_i \subseteq A(G_i)$. By (F2) and (F3) for w the empty sequence, it follows that $g_i \in A(G_i)$. To prove (QL) we show instead the equivalent

$$a \in A(U) \Longrightarrow a \wedge b \subseteq A(U \wedge b). \tag{11}$$

For this it is enough, by the fact that $A(U)$ is *inductive*, i.e. is the smallest satisfying (F1)–(F3), to prove that

$$W = \{a \in X : (\forall b \in X) a \wedge b \subseteq A(U \wedge b)\}$$

satisfies (F1)–(F3). We leave this to the reader.

To prove transitivity suppose $a \in A(U)$ and $U \subseteq A(V)$. Thus it is clear that $A(V)$ satisfies (F1) for U and also (F2), (F3). Hence since $A(U)$ is inductive, $A(U) \subseteq A(V)$. Thus $a \in A(V)$.

Finally, to see that $\lhd$ is the smallest cover relation satisfying the conditions of the theorem, suppose that $\lhd'$ is another. Let $A'(U) = \{a \in C : a \lhd' U\}$. Then (F1) and (F2) are clearly satisfied. To check (F3) one uses localisation repeatedly. Thus by the inductiveness of $A(U)$ we get $A(U) \subseteq A'(U)$, and indeed

$$a \lhd U \Longrightarrow a \lhd' U. \quad \square$$

5.1. *Transitivity as an inductive generation rule*

Here another direct method is presented. We propose here to look first at $\mathcal{U}$-formal topologies. Here the middle term U of

$$\frac{x \lhd U \qquad U \lhd V}{x \lhd V}$$

varies over $\mathcal{R}(X) = \mathcal{R}_{\mathcal{U}}(X)$. We may then obtain $\lhd$ as the least pre-fixed point of an operator

$$\Gamma : \mathcal{P}(X \times \mathcal{R}(X)) \to \mathcal{P}(X \times \mathcal{R}(X)).$$

Let $\mathcal{G} = (g_i, G_i)_{i:I}$, where $g_i : X$ and $G_i \subseteq X$. The triple $(X, \leq, \mathcal{G})$ is called a *presentation* of a formal topology [23]. The presentation is *$\mathcal{U}$-small*, if the preorder $(X, =_X, \leq)$ is $\mathcal{U}$-small, the preset I is contained in $\mathcal{U}$, and each $G_i \subseteq X$ $(i : I)$ is represented in $\mathcal{U}$. Take $\mathcal{R}(X) = \mathcal{R}_{\mathcal{U}}(X)$. For $\lhd \subseteq X \times \mathcal{R}(X)$ define $\lhd' = \Gamma(\lhd)$ by letting $a \lhd' W$ if, and only if, at least one of the following (C1–C5) holds:

(C1) $(\exists i : I)(a =_X g_i \;\&\; W = G_i)$,
(C2) $a \in W$,
(C3) $(\exists U : \mathcal{R}(X))\,(a \lhd U \;\&\; U \lhd W)$,
(C4) $(\exists U, V : \mathcal{R}(X))\,(a \lhd U \;\&\; a \lhd V \;\&\; W = U \wedge V)$,
(C5) $(\exists b : X)\,(a \leq b \;\&\; W = \{b\})$.

These clauses correspond respectively to (GE), (R), (T), (QL) and (QE). The main lemma for this operator is:

Lemma 5.2. *Let $(X, \leq, \mathcal{G})$ be a $\mathcal{U}$-small presentation of a formal space. There exists a cover relation $\lhd_m \subseteq X \times \mathcal{R}_{\mathcal{U}}(X)$ with*

(i) *$\Gamma(\lhd_m) \subseteq \lhd_m$, and such that*
(ii) *for any $\lhd \subseteq X \times \mathcal{R}_{\mathcal{U}}(X)$ with $\Gamma(\lhd) \subseteq \lhd$, we have $\lhd_m \subseteq \lhd$.*

Proof. See below.

This lemma states that $\lhd_m$ is a minimal pre-fixed point of the operator. It is the *minimal cover generated* by $\mathcal{G}$, and $(X, \leq, \lhd_m)$ is a $\mathcal{U}$-formal space. Consider now the operator E which is like Γ except that $\mathcal{R}(X)$ in (C3) and (C4) is replaced by $\mathcal{P}(X)$, and $\lhd \subseteq X \times \mathcal{P}(X)$. Now $E(\lhd)$, by itself, contains an inadmissible existential quantification over $\mathcal{P}(X)$. We will however only use this expression in the context $E(\lhd) \subseteq \lhd$, in which case the existential variables can be eliminated, and replaced by universal quantification outside the implication as follows: For all $W, U, V : \mathcal{P}(X)$

$$\psi(a, W) \vee (a \lhd U \;\&\; U \lhd W) \vee (a \lhd U \;\&\; a \lhd V \;\&\; W = U \wedge V) \Rightarrow a \lhd W.$$

Here $\psi(a, W)$ is the disjunction of (C1), (C2) and (C5).

Theorem 5.3. *Let $(X, \leq, \mathcal{G})$ be a $\mathcal{U}$-small presentation for a formal space, and let $\lhd_m$ be the minimal cover generated by the presentation. The extension $\lhd_m^*$ is a minimal pre-fixed point of E. Consequently $\lhd_m^*$ is the smallest cover relation containing the axioms (GE) so that $(X, \leq, \lhd_m^*)$ is a formal topology.*

Proof. By Theorem 4.2 $(X, \leq, \lhd_m^*)$ is a formal topology, under the condition that $(X, \leq, \lhd_m)$ is a $\mathcal{U}$-formal topology. But the first part of Lemma 5.2 implies this condition, and moreover $g_i \lhd_m^* G_i$ for any $i : I$. Hence $E(\lhd_m^*) \subseteq \lhd_m^*$.

Suppose now that $E(\lhd) \subseteq \lhd$. Define the restriction $\lhd_R$ of $\lhd$ to $X \times \mathcal{R}(X)$

$$a \lhd_R U \iff_{\text{def}} a \lhd U.$$

It is now straightforward to check that

$$\Gamma(\lhd_R) \subseteq \lhd_R.$$

To see that clause (C1) is valid, remember that $G_i : \mathcal{R}(X)$. Now by the minimality of $\lhd_m$, we have $\lhd_m \subseteq \lhd_R$.

We claim that $\lhd^*_m \subseteq \lhd$: Suppose that $a \lhd^*_m U$. Then there is $U' : \mathcal{R}(X)$ with $a \lhd_m U' \subseteq U$. By the above $a \lhd_R U'$, and so by the fact that $\lhd_R$ is obtained by restriction $a \lhd U'$. The transitivity of $\lhd$ yields $a \lhd U$. This proves the desired inclusion. □

Proof of Lemma 5.2. By expanding the definition of $U \lhd W$ in (C3) it becomes evident that $\lhd$ occurs only strictly positively in the clauses (C1)–(C5). The same is true for the positivity clause (C6). There are standard techniques for finding the minimal pre-fixed point of strictly positive operators [1]; see also [16]. Even Grothendieck formal spaces can be dealt with in this way since $\lhd$ occurs strictly positively when (G1)–(G3) are expressed in terms of an operator. It is easy to generalise this to sites, where the preorder is replaced by a category.

We shall here indicate a different construction using so-called inductive-recursive definitions. The idea is to define three entities

$$D : \mathrm{Set} \qquad c : D \to \mathrm{Set} \qquad C : D \to \mathcal{R}(X)$$

by simultaneous induction and recursion. Each element d in D is thought of as an abstract derivation for a proposition of the form $a \lhd U$, where $a = c(d)$ and $U = C(d)$. For any $U : \mathcal{R}(X)$ let $U^\dagger$ denote the preset $(\Sigma y : X)(y \in U)$ and for its elements x let x_1 denote its first component. The following introduction rules are assumed:

$$\frac{a : X \quad U : \mathcal{R}(X) \quad r : (a \in U)}{\mathrm{ref}(a, U, r) : D} \qquad \begin{aligned} c(\mathrm{ref}(a, U, r)) &= a \\ C(\mathrm{ref}(a, U, r)) &= U \end{aligned} \tag{12}$$

$$\frac{V : \mathcal{R}(X) \quad p : D \quad q : C(p)^\dagger \to D \quad r : (\forall x : C(p)^\dagger)\, c(qx) =_X x_1 \,\&\, C(qx) = V}{\mathrm{trans}(V, p, q, r) : D} \tag{13}$$

$$\begin{aligned} c(\mathrm{trans}(V, p, q, r)) &= c(p) \\ C(\mathrm{trans}(V, p, q, r)) &= V \end{aligned}$$

$$\frac{p : D \quad q : D \quad r : c(p) =_X c(q)}{\mathrm{loc}(p, q, r) : D} \qquad \begin{aligned} c(\mathrm{loc}(p, q, r)) &= c(p) \\ C(\mathrm{loc}(p, q, r)) &= C(p) \wedge C(q) \end{aligned} \tag{14}$$

$$\frac{a : X \quad b : X \quad p : (a \leq b)}{\mathrm{ext}(a, b, p) : D} \qquad \begin{aligned} c(\mathrm{ext}(a, b, p)) &= a \\ C(\mathrm{ext}(a, b, p)) &= \{b\} \end{aligned} \tag{15}$$

$$\frac{i : I}{\mathrm{axiom}(i) : D} \qquad \begin{aligned} c(\mathrm{axiom}(i)) &= g_i \\ C(\mathrm{axiom}(i)) &= G_i. \end{aligned} \tag{16}$$

Define $\lhd \subseteq X \times \mathcal{R}(X)$ by

$$a \lhd U \iff_{\mathrm{def}} (\exists d : D)\,(a =_X c(d) \,\&\, C(d) = U).$$

The relation is then proven to be minimal using a straightforward induction on D. □

Remark 5.4. The inductive-recursive definition principle, which underlies type universes and similar structures, was systematised by [5].

5.2. The positivity predicate

The positivity predicate, which is used to affirmatively state that a basic neighbourhood is not covered by an empty set, occurs first in locale theory [8]. There it has a neat impredicative definition, as in Theorem 5.5(ii) below. It was realized by Martin-Löf that in the predicative setting it could be regarded as coinductively defined. Coquand [3] gives a construction involving only a countable chain of sets. Theorem 5.7 is inspired by that construction.

We suggest here an easy way to handle the positivity predicate in the present context. Let $(X, \leq)$ be a preorder. Suppose that $\mathcal{G} = (g_i, G_i)_{i:I}$ is a family of basic covers. Let $P \subseteq X$ be a fixed set. Suppose we want to find the smallest formal topology including the generating cover and satisfying the *positivity condition*

$$(a \in P \Rightarrow a \lhd U) \Longrightarrow a \lhd U.$$

A *positivity predicate* $P \subseteq X$ should in addition satisfy the *monotonicity condition*

$$\text{(M)} \qquad \frac{a \in P \qquad a \lhd U}{U \cap P \text{ inhabited}}.$$

For the remainder of this section let $\mathcal{U}$ be a regular universe which is large enough for the presentation $(X, \leq; \mathcal{G})$ to be $\mathcal{U}$-small. We extend operator Γ to Γ_P by adding

(C6) $a \in P \Longrightarrow a \lhd W$

as a new disjunct. It purpose is to ensure that the inductively defined cover relation satisfies the positivity condition. E is similarly extended to E_P. Lemma 5.2 extends to Γ_P. Let $\lhd_P$ be the associated minimal pre-fixed point.

A subset $P \subseteq X$ is said to be *sympathetic* to $(X, \leq, \mathcal{G})$ if the following monotonicity conditions hold:

(M1) $a \in P$, $a \leq b$ implies $b \in P$,
(M2) $a \in P$, $a \leq g_i$ implies $(G_i \cdot a) \cap P$ inhabited

(cf. [4]).

The empty set is sympathetic to any X, which gives no positive elements. Note also that if P satisfies (M2) then so does the subset

$$P^{\uparrow} = \{x \in X : (\exists a \in P)\, a \leq x\},$$

which automatically satisfies (M1).

Theorem 5.5. *Let $(X, \leq, \mathcal{G})$ be a $\mathcal{U}$-small presentation for a formal space. Let $\mathcal{R}(X) = \mathcal{R}_{\mathcal{U}}(X)$. Suppose that $P \subseteq X$ is sympathetic to this presentation. Let Γ_P be the corresponding operator, and let $\lhd_P \subseteq X \times \mathcal{R}(X)$ be the minimal pre-fixed point. Then:*

(i) *The monotonicity property*

$$\text{(M)} \qquad \frac{a \in P \qquad a \lhd_P U}{U \cap P \text{ inhabited}}$$

holds for $U : \mathcal{R}(X)$.

(ii) *If* $P : \mathcal{R}(X)$, *then it must satisfy*

$$P = \{x \in X : (\forall U : \mathcal{R}(X))\,(x \lhd_P U \Rightarrow U \text{ inhabited })\}.$$

Proof. Part (i): If $P \cap V$ is inhabited, we say that V *positive* or write $\mathrm{Pos}(V)$. Define a relation $\lhd_r \subseteq X \times \mathcal{R}(X)$ by

$$a \lhd_r U \iff_{\text{def}} (\forall b : X)[\mathrm{Pos}(a \cdot b) \Rightarrow \mathrm{Pos}(U \cdot b)].$$

Here $a \cdot b$ is the restriction $\{a\} \cdot b$. We prove that this is a cover relation satisfying positivity and (GE), i.e. that $\Gamma_P(\lhd_r) \subseteq \lhd_r$.

(C1): The proof of $g_i \lhd_r G_i$ is an application of (M2), for if $c \leq g_i, b$ and c is positive then so is $G_i \cdot c$, and hence also $G_i \cdot b$.

(C2): This is immediate by definition.

(C3): Suppose $a \lhd_r U$, $U \lhd_r V$ and $\mathrm{Pos}(a \cdot b)$. Then $\mathrm{Pos}(U \cdot b)$, so for some $a' \in U$, $\mathrm{Pos}(a' \cdot b)$. Hence also $\mathrm{Pos}(V \cdot b)$.

(C4): Suppose $a \lhd_r U$ and $a \lhd_r V$. Assume that $a \cdot b$ is positive. Thus there is $c \in P$ with $c \leq a, b$. Hence $a \cdot c$ is positive, so $U \cdot c$ is positive. This gives $d \in U \cdot c$ such that $d \in P$. Since $d \leq c \leq a$, $\mathrm{Pos}(a \cdot d)$. Thereby $V \cdot d$ must be positive, and thus contain an element $e \in P$ such that $e \in U \wedge V \wedge \{a\}$.

(C5): This is immediate by definition.

(C6): Suppose we know the implication $\mathrm{Pos}(a) \Rightarrow a \lhd_r U$. From $\mathrm{Pos}(a \cdot b)$ it follows that there is $c \in P$ with $c \leq a$ and $c \leq b$. Property (M1) entails $a \in P$, so the implication gives $a \lhd_r U$. Now since $\mathrm{Pos}(a \cdot b)$, this yields $\mathrm{Pos}(U \cdot b)$. But b was arbitrary, so we conclude that $a \lhd_r U$.

By the extended version of Lemma 5.2, we get $\lhd_P \subseteq \lhd_r$. Suppose $a \lhd_P U$. From $a \in P$ it follows trivially that $\mathrm{Pos}(a \cdot a)$. Since $a \lhd_r U$, we thus have $\mathrm{Pos}(U \cdot a)$. In particular, there is $x \in U$ and $c \leq x$ such that $c \in P$. By (M1) we have $x \in P$, so $U \cap P$ is indeed inhabited. This proves (M).

Part (ii): Suppose $P : \mathcal{R}(X)$. (M) implies directly that any $x \in P$ is in the displayed set. Conversely, assume that x is in this set. Then $\{x\} \cap P : \mathcal{R}(X)$. Now

$$x \in P \Longrightarrow x \lhd_P \{x\} \cap P.$$

But $\lhd_P$ satisfies the positivity condition, so we get rid of the proviso $x \in P$, and have simply $x \lhd_P \{x\} \cap P$. By the assumption on x, the set $\{x\} \cap P$ is inhabited, so $x \in P$. $\square$

The following is now an extension of Theorem 5.3.

Theorem 5.6. *Let* $(X, \leq, \mathcal{G})$ *be a* $\mathcal{U}$*-small presentation for a formal space, and let* $\mathcal{R}(X) = \mathcal{R}_{\mathcal{U}}(X)$. *Suppose that* $P : \mathcal{R}(X)$ *is sympathetic to the presentation, and let* $\lhd_P \subseteq X \times \mathcal{R}(X)$ *be the minimal cover generated by the presentation and closed under the positivity rule. Then the extension* $\lhd_P^*$ *is a minimal pre-fixed point of* E_P, *and it satisfies (M) for any* U. *Consequently* $(X, \leq, \lhd_P^*, P)$ *is a formal topology with a positivity* P, *which is unique in* $\mathcal{R}(X)$.

Proof. $(X, \leq, \lhd_P)$ is by construction a formal topology with positivity candidate P. Note that since $P : \mathcal{R}(X)$, Theorem 4.2(iii) yields that P is a positivity candidate also for the extended version X^*. Hence $E_P(\lhd^*_P) \subseteq \lhd^*_P$ (using also Theorem 5.3). Now suppose that $\lhd \subseteq X \times \mathcal{P}(X)$ satisfies $E_P(\lhd) \subseteq \lhd$. Consider its restriction to $X \times \mathcal{R}(X)$:

$$a \lhd_R U \Longleftrightarrow_{\text{def}} a \lhd U.$$

As before the clauses (C1)–(C5) are valid for Γ_P since it is a restriction. From $E_P(\lhd) \subseteq \lhd$ follows then also (C6), so we have $\Gamma_P(\lhd_R) \subseteq \lhd_R$. By minimality, we have $\lhd_P \subseteq \lhd_R$. Now if $a \lhd^*_P U$, there is some $U' \subseteq U$ with $a \lhd_P U'$, and hence $a \lhd_R U'$, i.e. $a \lhd U'$. Hence $a \lhd U$ by transitivity of $\lhd$. We have shown $\lhd^*_P \subseteq \lhd$.

It is clear that the property (M) lifts from $\lhd_P$ to $\lhd^*_P$.

The last statement follows from Theorem 5.5. $\square$

The question remains: are there any interesting sympathetic sets? Consider an arbitrary regular universe $\mathcal{U}_0$. Let $P_0 = H(\mathcal{U}_0)$ be the union

$$\bigcup_{T:\mathcal{R}_{\mathcal{U}_0}(X)} \{T : (\forall a : X)(\forall i : I)(a \leq g_i, a \in T \Rightarrow (G_i \cdot a) \cap T \text{ inhabited})\}.$$

Suppose $a \in P_0$ and $a \leq g_i$. Thus for some $T \subseteq P_0$ we have $a \in T$ and moreover $(G_i \cdot a) \cap T$ is inhabited. Hence $(G_i \cdot a) \cap P_0$ is also inhabited. Consequently, P_0 satisfies (M2), and thus $P_0^{\uparrow}$ is sympathetic to $(X, \leq, \mathcal{G})$.

Theorem 5.7. *Let $(X, \leq, \mathcal{G})$, where $\mathcal{G} = (g_i, G_i)_{i:I}$. Let $\mathcal{U}_0$ be a regular universe so large that it contains I, the natural numbers $\mathbb{N}$ and that $(X, \leq)$ is $\mathcal{U}_0$-small. Then: $H(\mathcal{U}_0)$ is the largest set sympathetic to $(X, \leq, \mathcal{G})$.*

Proof. By the argument above $H(\mathcal{U}_0)^{\uparrow}$ is clearly sympathetic to $(X, \leq, \mathcal{G})$. Suppose that $M \subseteq X$ is also sympathetic, so that

$$(\forall a : X)(\forall i : I)(a \leq g_i \;\&\; a \in M \Rightarrow (G_i \cdot a) \cap M \text{ inhabited}). \tag{17}$$

Let $x \in M$. We construct a sequence $M_0, M_1, M_2, \ldots \subseteq M$ with $x \in M_0$ and such that for all $k = 0, 1, 2, \ldots$

$$(\forall a : X)(\forall i : I)(a \leq g_i \;\&\; a \in M_k \Rightarrow (G_i \cdot a) \cap M_{k+1} \text{ inhabited}).$$

Moreover, we make sure that $M_k : \mathcal{R}_{\mathcal{U}_0}(X)$. By the assumption that $\mathbb{N}$ is contained in $\mathcal{U}_0$, we have that

$$M_\omega = \bigcup_{k:\mathbb{N}} M_k$$

belongs to the same regular power set. It is now easy to check that M_ω satisfies (17). Hence $x \in H(\mathcal{U}_0)$, by letting $T = M_\omega$. Thus we have the inclusion $M \subseteq H(\mathcal{U}_0)$. It follows that $H(\mathcal{U}_0)$ is the largest subset of X satisfying (M2). But letting $M = H(\mathcal{U}_0)^{\uparrow}$, we see that indeed $H(\mathcal{U}_0) = H(\mathcal{U}_0)^{\uparrow}$ so (M1) is satisfied as well.

It remains to construct the sequence. We do this using dependent choices. Suppose that $J : \mathcal{R}_{\mathcal{U}_0}(X)$ with $J \subseteq M$. Then clearly

$$(\forall a : X)(\forall i : I)(a \leq g_i \;\&\; a \in J \Rightarrow (G_i \cdot a) \cap M \text{ inhabited}). \tag{18}$$

Form the *preset*

$$K(J) = (\Sigma a : X)(\Sigma i : I)(a \leq g_i \ \& \ a \in J).$$

(Note that $K(J)$ is in $\mathcal{U}_0$, since $J : \mathcal{R}_{\mathcal{U}_0}(X)$.) Then (18) may rewritten as

$$(\forall (a, i, p) : K(J))(\exists b : X)(b \in (G_i \cdot a) \cap M).$$

Use type-theoretic choice to find $f : K(J) \to X$ with

$$(\forall (a, i, p) : K(J))(f(a, i, p) \in (G_i \cdot a) \cap M).$$

Now let

$$J' = \bigcup_{w:K(J)} \{f(w)\}.$$

We have thus

$$(\forall a : X)(\forall i : I)(a \leq g_i \ \& \ a \in J \Rightarrow (G_i \cdot a) \cap J' \text{ inhabited}). \tag{19}$$

The dependent choice principle thus gives the desired sequence, with $M_0 = \{x\}$, and $M_{k+1} = J'$ if $M_k = J$. $\square$

Let $P = H(\mathcal{U}_0)$, where $\mathcal{U}_0$ is as in Theorem 5.6. Let $\mathcal{U}_1$ be a regular universe that makes $(X, \leq, \mathcal{G}, P)$ $\mathcal{U}_1$-small. Then P and $\mathcal{U}_1$ satisfy the conditions of Theorem 5.5, so we have constructed a canonical positivity for the minimal extension X^*.

Further reading

See [6].

Acknowledgements

I am grateful to Thierry Coquand for discussions and the opportunity to present a preliminary version of this work at a seminar in Göteborg. I thank the referee for suggesting the simplified existence proof of inductively defined covers (Theorem 5.1).

The author is supported by a grant from the Swedish Research Council (VR).

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